Journal of Mathematical Physics, Analysis, Geometry
20XX, Vol. XX, No. X, pp. 1–10
doi:

On the number of zeros of functions in analytic quasianalytic classes

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A space of analytic functions in the unit disc with uniformly continuous derivatives is said to be quasianalytic if the boundary value of a non-zero function from the class can not have a zero of infinite multiplicity. Such classes were described in the 1950-s and 1960-s by Carleson, Rodrigues-Salinas and Korenblum. A non-zero function from a quasianalytic space of analytic functions can only have a finite number of zeros in the closed disc. Recently, Borichev, Frank, and Volberg proved an explicit estimate on the number of zeros, for the case of quasianalytic Gevrey classes. Here, an estimate of similar form for general analytic quasianalytic classes is proved using a reduction to the classical quasianalyticity problem.

Key words: Quasianalytic class, analytic quasianalyticity, number of zeros

Mathematical Subject Classification 2010: 26E10, 30D60, 30H99

1. Introduction

Analytic quasianalyticity.. Let $W = (w_n)_{n=0}^\infty$ be a weight such that

$$w_n \in [1, +\infty], \quad \sum_{n=0}^\infty \frac{1}{w_n} = 1 \ , \quad \frac{1}{w_n} = O(n^{-\infty}) \ . \tag{1.1}$$

Consider the following space $\mathfrak{A}_W$ of analytic functions in the unit disc $\mathbb{D} = \{|z| < 1\}$:

$$\mathfrak{A}_W = \left\{ f(z) = \sum_{n=0}^\infty a_n z^n \mid \|f\|_W \stackrel{\text{def}}{=} \sup_n |a_n| w_n < \infty \right\} \ . \tag{1.2}$$

For each k, the k-th derivative $f^{(k)}$ of a function $f \in \mathfrak{A}_W$ is uniformly continuous in $\mathbb{D}$, and hence admits boundary values

$$f^{(k)}(e^{i\theta}) = \lim_{z \to e^{i\theta}, \, z \in \mathbb{D}} f^{(k)}(z)$$

on $\partial\mathbb{D}$.

The class $\mathfrak{A}_W$ is said to be quasianalytic if a non-zero function $f \in \mathfrak{A}_W$ can not vanish with all derivatives at a point:

$$\forall k \geq 0 \, f^{(k)}(e^{i\theta_0}) = 0 \implies f \equiv 0 \ , \text{ i.e. } \forall n \geq 0 \, a_n = 0 \ . \tag{1.3}$$

A result proved by Carleson [5], Rodrigues-Salinas [14] and Korenblum [8] (which we state explicitly in Remark 1.5 at the end of this introduction) implies that the condition

$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} = \infty , \quad \text{where } M_k = \sum_{n=0}^{\infty} \frac{n^{k/2}}{w_n} , \tag{1.4}$$

is sufficient for quasianalyticity. If the weights are sufficiently regular, e.g. $w_{n-1} \leq w_n$ and $w_{2n} \leq \sqrt{w_n w_{4n}}$ for $n \geq 1$, the condition (1.4) is also necessary for (1.3). [In general, the condition (1.4) is not necessary. To ensure the quasianalyticity of the class $\mathfrak{A}_W$, it suffices for the measure $\sum_{n=0}^{\infty} w_n^{-1}\delta_n$ to be Stieltjes-determinate; this condition is strictly weaker than (1.4).] For such regular weights, the condition (1.4) is equivalent to the divergence

$$\sum_{n \geq 0} \frac{\log w_n}{1 + n^{3/2}} = \infty . \tag{1.5}$$

For example, the Gevrey weights

$$W^{(\alpha,a)} = (w_n^{(\alpha,a)})_{n\geq 0} , \quad w_n^{(\alpha,a)} = \exp(an^\alpha + c(\alpha, a)) , \tag{1.6}$$

where $c(\alpha, a) = \log \sum_{n\geq 0} \exp(-an^\alpha)$ is determined by the normalisation (1.1), define a quasianalytic class if and only if $\alpha \geq 1/2$.

More recently, the problem of analytic quasianalyticity (for the classes $\mathfrak{D}_M \supset \mathfrak{A}_W$ as in Remark 1.5 below) was studied by Borichev [3], who obtained a new proof of quasianalyticity in the quasianalytic case (1.4) as well as a bound on the growth of f near a zero of infinite multiplicity in the case when (1.4) fails.

Zeros in the closed disc, and an application in spectral theory.. If the space $\mathfrak{A}_W$ is quasianalytic, a non-zero function $f \in \mathfrak{A}_W$ has a finite number of zeros in $\overline{\mathbb{D}}$, counting multiplicity. Indeed, if f has an infinite number of zeros, these have an accumulation point $e^{i\theta_0} \in \partial\mathbb{D}$, and then f vanishes with all derivatives at $e^{i\theta_0}$.

This fact was exploited by Pavlov [11, 12] to show that a non-selfadjoint Schrödinger operator $Hy = -y'' + q(x)y$ with a continuous complex potential $q : \mathbb{R}_+ \to \mathbb{C}$, defined on the semiaxis $[0, \infty)$ with the boundary condition $y(0) - hy'(0) = 0$, has a finite number of eigenvalues, counting multiplicity, if

$$b_k = \int_0^\infty |q(x)|x^{k+1}dx < \infty \text{ for } k \geq 0 \text{ and } \int_0 \log \inf_k \frac{(\frac{b_{k+1}}{k+1} + b_k)t^k}{k!} dt = -\infty . \tag{1.7}$$

For example, the condition $|q(x)| \leq C \exp(-cx^\alpha)$ implies (1.7) if and only if $\alpha \geq \frac{1}{2}$. For $\alpha < \frac{1}{2}$ Pavlov constructed a potential such $|q(x)| \leq C \exp(-cx^\alpha)$ but H has infinitely many eigenvalues.

Recently, Bairamov, Çakar and Krall [1] and Golinskii and Egorova [7] obtained counterparts of Pavlov's results for non-selfadjoint Jacobi matrices. Consider the operator J acting on $\ell_2(\mathbb{Z}_+)$ via

$$(Jy)(n) = a_n y(n+1) + b_n y(n) + \mathbb{1}_{n\geq 1} c_{n-1} y(n-1) , \quad n \geq 0 . \tag{1.8}$$

It follows from the results of [7] that if

$$\sum_{k=1}^{\infty} m_k^{-1/k} = \infty \ , \quad \text{where} \quad m_k = \sum_{n=0}^{\infty} (|b_n| + |a_n c_n - 1|)\, n^{k/2} \ , \tag{1.9}$$

then J has a finite number of eigenvalues, counting multiplicity. The condition (1.9) holds, for example, when

$$|b_n| + |a_n c_n - 1| \leq C \exp(-cn^{\alpha})$$

with $\alpha \geq 1/2$, whereas for $\alpha < 1/2$ there exists [7] such an operator with infinitely many eigenvalues.

Estimates on the number of zeros.. Denote by n_f the number of zeros of f in $\overline{\mathbb{D}}$, counting multiplicity, and let

$$N_W(A) = \sup \left\{ n_f \,\middle|\, f \in \mathfrak{A}_W \ , \ |f(0)| \geq e^{-A} \|f\|_W \right\} \ , \quad A \geq 0 \ . \tag{1.10}$$

A compactness argument shows that $N_W(A)$ is finite for any $A < \infty$. However, it is also of interest to obtain explicit bound on N_W, and in particular to investigate the asymptotic behaviour as $A \to +\infty$. Using the method of Pavlov [11,12], such bounds can be translated into explicit bounds on the number of eigenvalues of the Schrödinger operator H as well as of its Jacobi counterpart J.

In view of these applications, Borichev, Frank and Volberg [4] proved an explicit bound on $N_W(A)$ for the Gevrey weights (1.6). Their results imply that

$$N_{W(\alpha,a)} \leq \begin{cases} C(\alpha, a) A^{\frac{\alpha}{2\alpha-1}} \ , & \alpha \in (\frac{1}{2}, \frac{1}{2} + \epsilon] \\ C_1(a) \exp(C_2(a)\sqrt{A}) \ , & \alpha = \frac{1}{2} \end{cases} \ , \tag{1.11}$$

with explicit C, C_1, C_2, along with improved bounds for small values of A. The argument of [4] is based on the method of pseudoanalytic extension introduced by Dyn′kin [6] and applied to analytic quasianalyticity by Borichev in [3].

Here we employ a reduction to the classical (Hadamard) quasianalyticity problem to prove

Proposition 1.1. *Let W be a weight as in (1.1) satisfying the condition (1.4), and let*

$$h(p) = \frac{M_{p-1}}{M_p} \ , \ p \geq 1; \qquad H(p) = \sum_{k=1}^{p} h(k) \ ; \tag{1.12}$$

$$h^{-1}(\epsilon) = \min \left\{ p \geq 1 \,\middle|\, h(p) \leq \epsilon \right\} \ , \quad H^{-1}(R) = \min \left\{ p \geq 0 \,\middle|\, H(p) \geq R \right\} \ . \tag{1.13}$$

Then the quantity $N_W(A)$ from (1.10) satisfies

$$N_W(A) \leq 300\, h \left(2 \max(p(A), h^{-1}(\frac{1}{p(A)})) \right)^2 \ , \quad \text{where } p(A) = H^{-1}(H(\lceil A + 3 \rceil) + 25\sqrt{A}) \ . \tag{1.14}$$

Remark 1.2. In our normalisation (1.1), $N_W(A) = 0$ for $A < \log 2$ as a consequence of the Rouché theorem, hence (1.14) is meaningful for $A \geq \log 2$.

Remark 1.3. In the Gevrey case (1.6),

$$h(p) \asymp p^{-\frac{1}{2\alpha}} \ , \quad H(p) \asymp \begin{cases} p^{1-\frac{1}{2\alpha}} \ , & \frac{1}{2} < \alpha \leq 1 \\ \log p \ , & p = \frac{1}{2} \end{cases} \ ,$$

hence the bound (1.14) implies that

$$N_{W(\alpha,a)}(A) \leq \begin{cases} C'(\alpha, a) A^{\frac{2\alpha}{2\alpha-1}} \ , & \alpha \in (\frac{1}{2}, 1] \\ C_1'(a) \exp(C_2'(a)\sqrt{A}) \ , & \alpha = \frac{1}{2} \end{cases} \ , \tag{1.15}$$

which is similar to (1.11), albeit with an inferior exponent for $\alpha > \frac{1}{2}$.

Remark 1.4. The estimate (1.14) remains valid in the non-quasianalytic situation, provided that A is sufficiently small for the right-hand side to be finite, i.e.

$$\sum_{k > \lceil A+3 \rceil} h(k) > 25\sqrt{A} \ . \tag{1.16}$$

Note that the condition (1.16) may hold for large A (particularly, for $A \geq \log 2$) if the series $\sum M_{k-1}/M_k$ converges slowly enough.

Remark 1.5. Proposition 1.1 also yields bound on the number of zeros of a function in the Carleson–Salinas–Korenblum class

$$\mathfrak{D}_M = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \,\middle|\, \|f\|_{\mathfrak{D}_M} \stackrel{\text{def}}{=} \sup_k \sup_{|z|<1} \frac{|f^{(k)}(z)|}{M_{2k}} < \infty \right\}$$

associated with a positive sequence $M = (M_k)_{k \geq 0}$. We sketch the (well-known) reduction: first, one may assume without loss of generality that $M_k \leq \sqrt{M_{k-1}M_{k+1}}$. The theorem of Carleson–Salinas–Korenblum asserts that in this case $\mathfrak{D}_M$ is quasianalytic if and only if

$$\sum_{k \geq 0} \frac{M_{k-1}}{M_k} = \infty \ . \tag{1.17}$$

Construct the weight $W(M) = (w_n)_{n \geq 0}$, where

$$w_n = \frac{\tilde{w}_n}{\sum_{m=0}^{\infty} \tilde{w}_m} \ , \quad \tilde{w}_m = \max_{0 \leq k \leq m} \frac{m(m-1)\cdots(m-k+1)}{M_{2k}}$$

so that $\mathfrak{D}_M \subset \mathfrak{A}_{W(M)}$. One can check that if (1.17) holds, then also

$$M^1 = (M_k^1)_{k \geq 0} \ , \quad M_k^1 = \sum_{n \geq 0} \frac{n^{k/2}}{w_n}$$

satisfies $\sum_{k \geq 0} M_{k-1}^1 / M_k^1 = \infty$. Therefore Proposition 1.1 applied to $W(M)$ yields an estimate on

$$N_{\mathfrak{D}_M}(A) = \sup \left\{ n_f \,\middle|\, f \in \mathfrak{D}_M \ , \ |f(0)| \geq e^{-A} \|f\|_{\mathfrak{D}_M} \right\} \ , \quad A \geq 0 \ , \tag{1.18}$$

for an arbitrary quasianalytic $\mathfrak{D}_M$.

2. Proof of Proposition 1.1

The proof is based on the following construction, similar to the one using which the determinacy criteria for the moment problem in the Stieltjes case are derived from those in the Hamburger case (see [15] for a further application of a similar construction). To every

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathfrak{A}_W$$

we associate a function

$$\phi_f(x) = \sum_{n=0}^{\infty} a_n \cos(\sqrt{n}x) \ , \quad x \in \mathbb{R} \ .$$

We have:

$$|\phi_f^{(k)}(x)| \leq \sum_{n=0}^{\infty} |a_n| n^{k/2} \leq \|f\|_W M_k \ ,$$

i.e. ϕ_f lies in the space

$$\mathfrak{Q}_M = \left\{ \phi \in C^\infty(\mathbb{R}) \,\Big|\, \|\phi\|_{\mathfrak{Q}_M} \overset{\mathrm{def}}{=} \sup_k \frac{\|\phi^{(k)}\|_\infty}{M_k} < \infty \right\}$$

defined by the sequence $M = (M_k)_{k \geq 0}$ of (1.4). According to the Denjoy–Carleman theorem in the form of Mandelbrojt (see [2] or [10], and also the comment following Lemma 2.3 below), the condition $\sum_{k=1}^{\infty} M_{k-1}/M_k = \infty$ implies that the class $\mathfrak{Q}_M$ is quasianalytic. [In our case, the sequence M is logarithmically convex, i.e. $M_k \leq \sqrt{M_{k+1}M_{k-1}}$ for $k \geq 1$, hence the condition $\sum_{k=1}^{\infty} M_{k-1}/M_k = \infty$ is necessary and sufficient for the quasianalyticity of $\mathfrak{Q}_M$.] This implies the sufficiency part of the Carleson–Salinas–Korenblum condition (1.4) for the quasianalyticity of $\mathfrak{A}_W$: indeed, if f vanishes with all derivatives at 1, then ϕ_f vanishes with all derivatives at 0, and hence ϕ_f and f are identically zero.

To prove Proposition 1.1, we make these considerations quantitative. The argument rests on two lemmas. The first one asserts that ϕ_f and its first few derivatives are small at 0 if f has many zeros near 1.

Lemma 2.1. *Let $\epsilon \in (0,1)$, and let m be the number of zeros of $f \in \mathfrak{A}_W$ in the domain $\{|z| \leq 1 \ , |z-1| < \epsilon\}$, counted with multiplicity. Then*

$$|\phi_f^{(2k)}(0)| \leq \left(\frac{4e\epsilon}{m}\right)^{m-k} M_{2m}\|f\|_W \ , \quad 0 \leq k \leq \min(\frac{m}{2}, \sqrt{\frac{m}{8\epsilon}}) \ .$$

The second lemma guarantees that there is a point not too far from 0 at which ϕ_f is not too small. The current version, with the sharp power of A, was kindly communicated by F. Nazarov.

Lemma 2.2. *Let $\phi(x) = \sum_{n=0}^{\infty} a_n \cos(\sqrt{n}x)$ be such that $|a_0| \geq e^{-A}$ and $\sum |a_n| \leq 1$. Then there exists $x \in [0, 9\sqrt{A}]$ such that $|\phi(x)| \geq e^{-A-3}$.*

To derive the proposition from the two lemmas, we use a propagation of smallness argument due to Bang [2], which we state as

Lemma 2.3. *Let $M = (M_k)_{k\geq 0}$ be a sequence of positive numbers such that $M_k \leq \sqrt{M_{k-1}M_{k+1}}$ for $k \geq 1$. For $\phi \in \mathfrak{Q}_M$, define a nested sequence of sets $\mathbb{R} = B_0(\phi) \supset B_1(\phi) \supset B_2(\phi) \supset \cdots$ via*

$$B_p(\phi) = \left\{ x \in \mathbb{R} \,\middle|\, \forall 0 \leq k < p \, |\phi^{(k)}(x)| \leq e^{k-p} M_k \|\phi\|_{\mathfrak{Q}_M} \right\} .$$

Then for $0 \leq q < p$

$$\mathrm{dist}(B_p(\phi), \mathbb{R} \setminus B_q(\phi)) \geq \frac{1}{e}(H(p) - H(q)) = \frac{1}{e} \sum_{k=q+1}^{p} \frac{M_{k-1}}{M_k} .$$

(As pointed out in [2], this lemma readily implies the Denjoy–Carleman theorem mentioned above.) The proofs of the lemmas are postponed to the next section, and we now proceed to

Proof of Proposition 1.1. Without loss of generality we may assume that $\|f\|_W = 1$, so that $\|\phi_f\|_{\mathfrak{Q}_M} \leq 1$. Denote by $n_f(S)$ the number of zeros of f in $S \subset \overline{\mathbb{D}}$, counting multiplicity. By Jensen's formula

$$\begin{aligned} -A = \log|f(0)| &= \sum_{f(z)=0} \log|z| + \int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \\ &\leq \sum_{f(z)=0} \mathbb{1}_{|z|<1-\frac{\epsilon}{2}} \log|z| \\ &\leq \log(1 - \frac{\epsilon}{2})\, n_f(\{|z| < 1 - \frac{\epsilon}{2}\}) , \end{aligned}$$

hence

$$n_f \leq \frac{2}{\epsilon} A + n_f(\{|z| \geq 1 - \frac{\epsilon}{2}\}) \leq \frac{1}{\epsilon}(2A + 8m_\epsilon) , \tag{2.1}$$

where

$$m_\epsilon = \sup_\theta n_f(\{|z| \leq 1 \ , \ |z - e^{i\theta}| < \epsilon\}) .$$

Without loss of generality the supremum in the definition of m_ϵ is achieved when $\theta = 0$.

Let $p(A) = H^{-1}(H(\lceil A + 3 \rceil) + 25\sqrt{A})$, and let

$$m = \max(p(A), h^{-1}(\frac{1}{p(A)})) , \quad \epsilon = \frac{1}{4e^2} \frac{m M_{2m-1}^2}{M_{2m}^2} ,$$

so that

$$\sqrt{\frac{m}{2\epsilon}} = \sqrt{2} e \frac{M_{2m}}{M_{2m-1}} \geq \frac{M_m}{M_{m-1}} = \frac{1}{h(m)} \geq p(A) .$$

Let us show that $m_\epsilon < m$. Assume the contrary. Observe that

$$\frac{p(A)}{2} \leq \min(\frac{m}{2}, \sqrt{\frac{m}{8\epsilon}}) ,$$

therefore Lemma 2.1 yields

$$|\phi_f^{(2k)}(0)| \le \left(\frac{4e\epsilon}{m}\right)^{m-k} M_{2m} \tag{2.2}$$

for $0 \le k \le p(A)/2$. Estimating

$$\frac{\left(\frac{4e\epsilon}{m}\right)^{l-1} M_{2l-2}}{\left(\frac{4e\epsilon}{m}\right)^{l} M_{2l}} = e\frac{M_{2l-2}}{M_{2l}}\frac{M_{2m}^2}{M_{2m-1}^2} \ge e\ , \quad 0 \le l \le m\ ,$$

we obtain that

$$\left(\frac{4e\epsilon}{m}\right)^{m} M_{2m} = \left(\frac{4e\epsilon}{m}\right)^{k} M_{2k} \prod_{l=k+1}^{m} \frac{\left(\frac{4e\epsilon}{m}\right)^{l} M_{2l}}{\left(\frac{4e\epsilon}{m}\right)^{l-1} M_{2l-2}} \le e^{-(m-k)} \left(\frac{4e\epsilon}{m}\right)^{k} M_{2k}\ .$$

Therefore (using that $3k \le 3p(A)/2 \le m + p(A)$)

$$|\phi_f^{(2k)}(0)| \le e^{-(m-k)} M_{2k} \le e^{-(2k-p(A))} M_{2k}\ .$$

Trivially, $\phi_f^{(2k+1)}(0) = 0$ for all k. Hence $0 \in B_{p(A)}(\phi_f)$. On the other hand, by Lemma 2.2,

$$\mathrm{dist}(0, \mathbb{R} \setminus B_{\lceil A+3 \rceil}(\phi_f)) \le 9e\sqrt{A}\ .$$

Applying Lemma 2.3, we deduce that

$$H(p(A)) - H(\lceil A+3 \rceil) \le 9e\sqrt{A} < 25\sqrt{A}\ ,$$

in contradiction with the definition of $p(A)$. This completes the proof of the estimate $m_\epsilon < m$.

Returning to (2.1) and recalling that $m \ge p(A) \ge A$, we obtain:

$$n_f \le \frac{1}{\epsilon}(2A + 8m_\epsilon) \le \frac{10m}{\epsilon} \le 300h(2m)^2\ . \qquad \square$$

3. Proofs of the lemmas

In the proof of the Lemma 2.1, we use the following lemma which is borrowed from the work of M. Lavie [9, Lemma 3].

Lemma 3.1. *Let $R \subset \mathbb{C}$ be a closed convex set of diameter δ. If $f(z)$ is analytic in R and vanishes at m points of R (counting multiplicity), then*

$$\max_{z\in R} |f^{(k)}(z)| \le \frac{\delta^{m-k}}{(m-k)!} \max_{z\in R} |f^{(m)}(z)|\ , \quad 0 \le k \le m\ . \tag{3.1}$$

In [9], this inequality is proved by induction, using the formula

$$\frac{d^k}{dz^k}\frac{f(z)}{\alpha - z} = (\alpha - z)^{-k-1} \int_\alpha^z \frac{f^{(k+1)}(\zeta)}{(\alpha - \zeta)^k} d\zeta\ ,$$

valid if $f(\alpha) = 0$. As mentioned in [9], (3.1) can also be proved using the Hermite formula for divided differences.

Remark 3.2. By an approximation argument, the conditions of the lemma can be relaxed as follows: (a) if $R = \overline{\mathrm{int}\, R}$, then the lemma remains valid if instead of assuming that f is analytic in R, we assume that f analytic in $\mathrm{int}\, R$ and that $f, f', f', \cdots, f^{(m)}$ are uniformly continuous in R. (b) If $R \subset \mathbb{R}$, it suffices to assume that $f \in C^m(R)$.

Proof of Lemma 2.1. Recall (see [13]) that the Stirling numbers of the second kind are defined via

$$\left\{ \begin{matrix} k \\ l \end{matrix} \right\} = \frac{1}{l!} \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} j^k ,$$

so that

$$n^k = \sum_{l=0}^{k} \left\{ \begin{matrix} k \\ l \end{matrix} \right\} n(n-1)\cdots(n-l+1) ,$$

and that

$$0 \leq \left\{ \begin{matrix} k \\ l \end{matrix} \right\} \leq \frac{1}{2} \binom{k}{l} l^{k-l} \leq \frac{1}{2} k^{2(k-l)} .$$

Then

$$\begin{aligned} |\phi_f^{(2k)}(0)| &= \left| \sum_{n \geq 0} a_n n^k \right| \leq \sum_{l=0}^{k} \left\{ \begin{matrix} k \\ l \end{matrix} \right\} \left| \sum_{n \geq 0} a_n n(n-1)\cdots(n-l+1) \right| \\ &= \sum_{l=0}^{k} \left\{ \begin{matrix} k \\ l \end{matrix} \right\} |f^{(l)}(1)| \leq \frac{1}{2} \sum_{l=0}^{k} k^{2(k-l)} |f^{(l)}(1)| . \end{aligned}$$

By Lemma 3.1 and the subsequent Remark 3.2, we have:

$$|f^{(l)}(1)| \leq \frac{(2\epsilon)^{m-l}}{(m-l)!} M_{2m} ,$$

hence

$$\begin{aligned} |\phi_f^{(2k)}(0)| &\leq \frac{1}{2} \sum_{l=0}^{k} k^{2(k-l)} \frac{(2\epsilon)^{m-l}}{(m-l)!} M_{2m} \\ &\leq \frac{1}{2} \frac{(2\epsilon)^{m-k}}{(m-k)!} M_{2m} \sum_{l=0}^{k} \left(\frac{2k^2\epsilon}{m-k} \right)^{k-l} \\ &= \frac{1}{2} \frac{(2\epsilon)^{m-k}}{(m-k)!} M_{2m} \sum_{l=0}^{k} \left(\frac{4k^2\epsilon}{m} \right)^{k-l} \leq \left(\frac{4e\epsilon}{m} \right)^{m-k} M_{2m} , \end{aligned}$$

provided that $m \geq \max(2k, 8k^2\epsilon)$. □

Proof of Lemma 2.2 (F. Nazarov). Define a sequence of independent random variables X_j so that $X_j \sim \mathrm{Unif}[-\frac{\pi}{\sqrt{j}}, \frac{\pi}{\sqrt{j}}]$, and let $S_N = X_1 + \cdots + X_N$. Then

$$|S_N| \leq \pi \sum_{j=1}^{N} \frac{1}{\sqrt{j}} \leq 2\pi\sqrt{N}$$

and

$$g_N(\xi) = \mathbb{E}\cos(\xi S_N) = \mathbb{E}\exp(i\xi S_N) = \prod_{j=1}^N \frac{\sin\frac{\pi\xi}{\sqrt{j}}}{\frac{\pi\xi}{\sqrt{j}}} .$$

Therefore

$$\mathbb{E}\phi(S_N) = \sum_{n\geq 0} a_n g_N(\sqrt{n}) = a_0 + \sum_{n \geq N+1} a_n g_N(\sqrt{n}) .$$

Now, for $\xi \geq \sqrt{N}$

$$|g_N(\xi)| \leq \prod_{j=1}^N \frac{\sqrt{j}}{\pi\xi} \leq \prod_{j=1}^N \frac{1}{\pi} = \pi^{-N} ,$$

therefore

$$|\mathbb{E}\phi(S_N)| \geq |a_0| - \sum_{n\geq N+1} \pi^{-N}|a_n| \geq e^{-A} - \pi^{-N} .$$

Letting $N = \lceil A \rceil$, we obtain that there exists $x \in [0, 2\pi\sqrt{\lceil A\rceil}]$ such that

$$|\phi(x)| \geq e^{-A}(1 - e/\pi) \geq e^{-A-3} .$$

For $A \geq \frac{1}{2}$, $2\pi\sqrt{\lceil A \rceil} \leq 9\sqrt{A}$, as claimed. For $A < \frac{1}{2}$,

$$|\phi(0)| \geq e^{-1/2} - (1 - e^{-1/2}) \geq e^{-3} \geq e^{-A-3} . \qquad \square$$

Proof of Lemma 2.3. We reproduce the original argument of Bang [2]. It suffices to show that if $x \in B_p(\phi)$ and $h = |y - x| \leq \frac{1}{e}\frac{M_{p-1}}{M_p}$, then $y \in B_{p-1}(\phi)$. Expanding ϕ in a Taylor series, we have for $0 \leq k < p-1$:

$$\begin{aligned} |\phi^{(k)}(y)| &\leq \sum_{j=0}^{p-k-1} |\phi^{(k+j)}(x)|\frac{h^j}{j!} + |\phi^{(k+j)}(y_1)|\frac{h^{p-k}}{(p-k)!} \\ &\leq \sum_{j=0}^{p-k} e^{k+j-p} M_{k+j} \|\phi\|_{\mathfrak{Q}_M} \frac{h^j}{j!} . \end{aligned}$$

Now we bound $M_{k+j} \leq M_k (M_p/M_{p-1})^j$ and obtain:

$$\begin{aligned} |\phi^{(k)}(y)| &\leq e^{k-p} M_k \|\phi\|_{\mathfrak{Q}_M} \sum_{j=0}^{p-k-1} \frac{e^j (M_p/M_{p-1})^j h^j}{j!} \\ &\leq e^{k-p} M_k \|\phi\|_{\mathfrak{Q}_M} e^{eh\frac{M_p}{M_{p-1}}} \leq e^{k-p+1} M_k \|\phi\|_{\mathfrak{Q}_M} . \square \end{aligned}$$

Acknowledgments. I am grateful to A. Borichev, M. Sodin, J. Stoyanov, and A. Volberg for helpful comments, and to F. Nazarov for explaining me the proof of the current Lemma 2.2.

Supports. This work is supported in part by the European Research Council starting grant 639305 (SPECTRUM) and by a Royal Society Wolfson Research Merit Award.

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Received Month XX, 20XX, revised Month XX, 20XX.

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